Riemann

More normal coords:

$$R_{ab}{}^{c}{}_{d} = \partial_a \Gamma_b{}^c{}_d - \partial_b \Gamma_a{}^c{}_d$$

$$\Gamma_b{}^c{}_d = \frac{1}{2} g^{ce} \big(g_{eb,d} + g_{ed,b} - g_{bd,e} \big)$$

$$R_{abcd} = \frac{1}{2}\left(\cancel{g_{cd,ab}} + g_{bc,ad} - g_{bd,ac} \right) - \frac{1}{2}\left(\cancel{g_{cd,ba}} + g_{ac,bd} - g_{ad,bc} \right)$$

Rel'n w/ rep'ns of S_4: in 2d rep

$$g_{\cdot\cdot,\cdot\cdot} \in \mathrm{Fix}((12))$$

Extra rel'n (torsion-free)

$$\left(1 + (123) + (123)^2\right) g_{\cdot\cdot,\cdot\cdot} = 0 \qquad \text{i.e.} \sum_{\sigma \in S_3} g_{\sigma \cdot \cdot, \cdot}$$

from geod eq'n $\partial_r = \operatorname{grad} r$

Gauss's lemma: $r\partial_r = \nabla r$ $\qquad r\,dr = \delta_{ij} x^i dx^j$

$$\langle x^i \partial_i, \partial_e \rangle = \langle g^{jk} \delta_{ij} x^i \partial_k, \partial_e \rangle$$

$$\boxed{x^i g_{ie} = \delta_{ie} x^i}$$

$$x^i g_{ie} = x^i \delta_{ie} + \underbrace{x^i a_{jkie} x^j x^k}_{=0}$$

$$\Longleftrightarrow \sum_{\sigma \in S_3} a_{\sigma(ijk)e} = 0$$

$$dr = x^i dx^i$$

$1-(12)$

$1-(34)$

$1+\sigma+\sigma^2$

~~alt~~

~~triv~~

$(12)(34)$ ~~std~~

~~std ⊗ alt~~

$$\frac{x^i \partial_i}{r} = \frac{g^{ij} x^j \partial_i}{r}$$

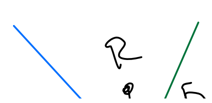
$$x^i = g^{ij} x^j = x^i - g_{ij,kl} x^k x^l x^j$$

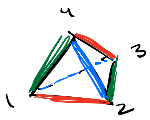
$\Rightarrow g_{\cdot\cdot,\cdot\cdot} \in$ Fun also

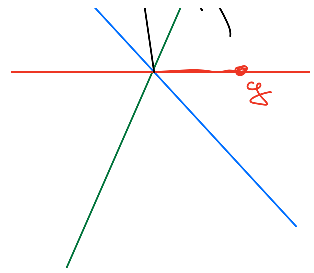
$\Rightarrow (13)(24)$ acts trivially

$$\Rightarrow R_{abcd} = g_{ad,cb} - g_{ac,bd}$$

$$R = \big((24) - (23) \big) g$$







Normal coordinate trick: 2^{nd} Bianchi identity

Use normal coords when you can to drop terms!

$$\nabla_a \nabla_b \nabla_c \partial_d - \nabla_b \nabla_a \nabla_c \partial_d = R(a,b)(\nabla_c \partial_d) = 0 \quad !$$

$2\,\cancel{3}\,1 - 1\,3\,2$

$1\,2\,3 - 2\,1\,3$

$3\,1\,2 - 3\,\cancel{2}\,1$

Recall sec determines R

Cor If sec $= C$ then

$$R(X,Y)Z = C(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$$

⌜ check Bianchi $\qquad \langle W, R(X,Y)Z\rangle \qquad \langle \cdot,\cdot\rangle$.

$$C(1+\sigma+\sigma^2)(1-C(23)) = C(1+\sigma+\sigma^2)(1-C(12)) = 0$$ ⌟

let
$$u_c(t) = \begin{cases} t & C=0 \\ R\sin\frac{t}{R} & C=\frac{1}{R^2}>0 \\ R\sinh\frac{t}{R} & C=-\frac{1}{R^2}<0 \end{cases}$$

Thm If (M,g) has const. sec. curvature C then it is locally isometric to $(dr^2 + u_c(r)^2 g_{S^{n-1}})$

⌜

compute dim 2

$$B = \tfrac{1}{2} g^{-1} L_{\partial_r} g$$

$$L_{\partial_r} B + B^2 = -R_{\partial_r} = R(\partial_r, X)\partial_r = -CI \quad \text{on } \partial_r^{\perp}$$

$$\boxed{R_{\partial_r} = -R(\partial_r, X)\partial_r = R(X,\partial_r)\partial_r}$$

Hyperbolic plane models

Wave/particle

Note geodesics are easier to find than distance functions.

in general. Family of geodesics that foliate!

so phrasing things in terms of Jacobi fields is a little more general.

Prop J is tangential to a variation thru geodesics iff

$$\nabla_t \nabla_t J = -R_{\dot\gamma}(J)$$

Rel'n to the Riccati eq'n

r picks out a Lagrangian subspace of the $2n$-dim'l space of Jacobi fields, namely $\{ J \text{ s.t. } \mathcal{L}_{\partial_r} J = 0 \}$

Rmk $\mathcal{L}_{\partial_r} J = 0$ makes sense

$$\mathfrak{X}(\gamma^* TM) = \mathrm{Der}(C(M), C(\mathbb{R})_\varphi)$$

$\partial_r J f - J \partial_r f$ makes sense b/c $\partial_r \in \mathrm{Der}(C(\mathbb{R}))$ also

$$\mathcal{L}_{\partial_r} J = \nabla_r J - \nabla_J \partial_r = \nabla_r J - B(J)$$

If $\nabla_r J = B(J)$ i.e. $\mathcal{L}_{\partial_r} J = 0$, then

$$(\mathcal{L}_{\partial_r} B)(J) + B^2 J = \nabla_r \nabla_r J$$

2^{nd} var formula + symp form. (7.)

Sol'ns w/ $J(0) = 0$ determined by $\nabla_r J(0) \in T_m$

$\longrightarrow$ metric on S^{n-1}